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Novel global stability criteria for high-order Hopfield-type neural networks with time-varying delays

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Abstract

This paper discusses a generalized model of high-order Hopfield-type neural networks with time-varying delays. Some novel global stability criteria of the system is derived by using Lyapunov method, linear matrix inequality (LMI) and analytic technique. The LMI-based criteria obtained here are computationally more flexible and more generic than many other existing criteria. A numerical example is given to illustrate our result.

Keywords: Stability; High-order Hopfield-type neural networks; Time-varying delays; Linear matrix inequality

1. Introduction

Hopfield [1,2] has proposed Hopfield neural networks (HNNs) which have found applications in a broad range of disciplines where the targeted problems can reduce to optimization problems. In some applications of HNNs such as associative memory and repetitive learning, many stability criteria have been derived in the literature with different delayed neural network models being considered [3–11], and based on linear matrix inequality (LMI), the authors obtained some sufficient conditions for delayed neural networks [15,16,26–30]. Since the existence of delays is frequently a source of instability for neural networks, there has been a considerable attention given in the literature on Hopfield-type neural networks with time delays. For example,

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see [12–17] and references therein. Higher-order neural networks have been investigated recently in [15–17]. It is known that high-order neural networks have stronger approximation property, faster convergence rate, greater storage capacity, and higher fault tolerance than lower-order neural networks.

As far as we know, few results have been reported in literature on the stability of high-order Hopfield-type neural networks with time-varying delays. In [16], by employing the LMI approach and the Lyapunov functional methods, several sufficient conditions are obtained for ensuring a class of high-order bidirectional associative memory (BAM) neural networks with constant time delay to be globally exponentially stable. By utilizing Lyapunov functional, Xu et al. [15] derived some sufficient conditions for the global asymptotic stability of the equilibrium point of a class of high-order Hopfield type neural networks with constant time delays in terms of LMI.

In this paper, we shall consider a class of such neural networks. By employing the LMI and Lyapunov functional methods, some less conservative sufficient conditions are obtained for ensuring the equilibrium point of the system to be globally asymptotically stable. The results extend and improve the earlier publications. An example is also worked through to illustrate our results.

2. System description and preliminaries

We consider the high-order Hopfield-type neural networks with time-varying delays described by

$$\begin{aligned}\frac{dy_i(t)}{dt} &= -c_i y_i(t) + \sum_{j=1}^{n} a_{ij} g_j\big(y_j(t)\big) + \sum_{j=1}^{n} b_{ij} g_j\big(y_j\big(t-\tau_j(t)\big)\big)\\ &\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} T_{ijl} g_j\big(y_j\big(t-\tau_j(t)\big)\big) g_l\big(y_l\big(t-\tau_l(t)\big)\big) + J_i,\end{aligned} \tag{1}$$

where $i \in \{1, 2, \dots, n\}$, $t \geqslant t_0$, $y_i(t)$ is the neuron state; c_i is positive constant, it denotes the rate with which the cell i resets its potential to the resting state; a_{ij}, b_{ij} are the first-order synaptic weights of the neural networks; T_{ijl} is the second-order synaptic weights of the neural networks; $\tau_j(t)$ $(j = 1, 2, \dots, n)$ is the transmission delay of the jth neuron such that $0 < \tau_j(t) \leqslant \tau^*$ and $\dot{\tau}_j(t) \leqslant \sigma < 1$, where τ^*, σ are constants; the activation function g_j is continuous on $[t_0 - \tau^*, +\infty)$; J_i is the external input.

Suppose that the system (1) is supplemented with initial conditions of the form

$$y_i(s) = \phi_i(s), \quad s \in [-\tau^*, 0],\ i = 1, 2, \dots, n, \tag{2}$$

where $\phi_i(s)$ $(i = 1, 2, \dots, n)$ are continuous on $[-\tau^*, 0] \times \Omega$.

Throughout this paper, we assume that there exists positive constants $L_i > 0$ and $\chi_i > 0$, $i = 1, 2, \dots, n$, such that the activation function g_i satisfies the following conditions:

(H_1) $$\big|g_i(u_i)\big| \leqslant \chi_i, \quad 0 < \frac{g_i(u_i) - g_i(v_i)}{u_i - v_i} \leqslant L_i,$$

where $u_i, v_i \in R$, $i = 1, 2, \dots, n$.

Due to the boundedness of the activation function g_i, by employing the well-known Brouwer's fixed point theorem, we can easily obtain that there exists an equilibrium point of the system (1).

The uniqueness of the equilibrium can be deduced from the asymptotic stability which will be proved subsequently.

Let y^* be an equilibrium point of (1), $y(t)$ be any solution of (1) and $x(t) = y(t) - y^*$. Set $f_j(x_j(t)) = g_j(y_j(t)) - g_j(y_j^*)$, $f_j(x_j(t - \tau_j(t))) = g_j(y_j(t - \tau_j(t))) - g_j(y_j^*)$. Apparently, for each $i = 1, 2, \dots, n$, we have

$$\left|f_j(z)\right| \leqslant L_j|z|, \quad \forall z \in R. \tag{3}$$

Substituting them into (1) and using Taylor's theorem, we can write (1) as

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j\big(x_j(t)\big) + \sum_{j=1}^{n} b_{ij} f_j\big(x_j\big(t - \tau_j(t)\big)\big) \\ &\quad + \sum_{j=1}^{n} \left(\sum_{l=1}^{n} (T_{ijl} + T_{ilj})\zeta_l \right) f_j\big(x_j\big(t - \tau_j(t)\big)\big), \end{aligned} \tag{4}$$

where $i = 1, 2, \dots, n$; ζ_l is between $g_l(y_l(t - \tau_l(t)))$ and $g_l(y_l^*)$.

Rewrite (4) into vector form as follows:

$$\frac{dx(t)}{dt} = -Cx(t) + Af\big(x(t)\big) + \big(B + \Gamma^T T_H\big) f\big(x\big(t - \tau(t)\big)\big), \tag{5}$$

where

$$\begin{aligned} &C = \operatorname{diag}(c_1, c_2, \dots, c_n), \\ &A = (a_{ij})_{n\times n}, \qquad B = (b_{ij})_{n\times n}, \\ &T_i = (T_{ijl})_{n\times n}, \\ &T_H = \big(T_1 + T_1^T, T_2 + T_2^T, \dots, T_n + T_n^T\big)^T, \\ &x\big(t - \tau(t)\big) = \big(x_1\big(t - \tau_1(t)\big), \dots, x_n\big(t - \tau_n(t)\big)\big)^T, \\ &f\big(x(t)\big) = \big(f_1\big(x_1(t)\big), \dots, f_n\big(x_n(t)\big)\big)^T, \\ &f\big(x\big(t - \tau(t)\big)\big) = \big(f_1\big(x_1\big(t - \tau_1(t)\big)\big), \dots, f_n\big(x_n\big(t - \tau_n(t)\big)\big)\big)^T, \\ &\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n)^T, \\ &\Gamma = \operatorname{diag}(\zeta, \zeta, \dots, \zeta). \end{aligned}$$

In order to obtain our results, we need establishing the following definitions and lemmas:

Lemma 1. [18] *Given any real matrices Σ_1, Σ_2, Σ_3 of appropriate dimensions and a scalar $\varepsilon > 0$ such that $0 < \Sigma_3 = \Sigma_3^T$. Then, the following inequality holds*:

$$\Sigma_1^T \Sigma_2 + \Sigma_2^T \Sigma_1 \leqslant \varepsilon \Sigma_1^T \Sigma_3 \Sigma_1 + \varepsilon^{-1} \Sigma_2^T \Sigma_3^{-1} \Sigma_2, \tag{6}$$

where the superscript T means the transpose of a matrix.

Lemma 2 *(Schur complement). Linear matrix inequality*:

$$\begin{pmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{pmatrix} > 0, \tag{7}$$

with $Q(x) = Q^T(x)$, $R(x) = R^T(x)$ and $S(x)$ depend affinity on x, is equivalent to

$$R(x) > 0, \qquad Q(x) - S(x)R^{-1}(x)S^T(x) > 0. \tag{8}$$

We also recall some basic facts about norms of vectors and matrices. Let $y = (y_1, y_2, \dots, y_n)^T \in R^n$. The three commonly used vector norms are: $\|y\|_1 = \sum_{i=1}^n |y_i|$, $\|y\|_2 = (\sum_{i=1}^n y_i^2)^{1/2}$, $\|y\|_\infty = \max_{1\leqslant i\leqslant n} |y_i|$. It is also known that $\|y\|_\infty \leqslant \|y\|_1$. The vector $|y|$ will denote $|y| = (|y_1|, |y_2|, \dots, |y_n|)^T$. For any matrix $V = (v_{ij})_{n\times n}$, $\lambda_m(V)$ and $\lambda_M(V)$ will denote the minimum and maximum eigenvalues of V, respectively. For the matrix V, $\|V\|_2^2 = \lambda_M(V^T V)$.

3. Stability criteria

In this section, we shall obtain some sufficient conditions for global asymptotic stability of the high-order Hopfield-type neural networks with time-varying delays.

If y^* is an equilibrium point of (1), then $x^* = 0$ is an equilibrium point of (4) and (5). To prove the global asymptotic stability of the equilibrium point of (1), it is sufficient to prove the global asymptotic stability of the trivial solution of (4) or (5).

Theorem 1. *The origin of system* (5) *is asymptotically stable if condition* (H_1) *is satisfied and moreover there exist positive definite symmetric matrices* P, Σ_1, Σ_2 *and a scalar* $\varepsilon > 0$ *such that*

$$\begin{aligned}\Omega &= PC + C^T P - \frac{1}{\varepsilon} PA\Sigma_1^{-1} A^T P - \varepsilon L\Sigma_1 L \\ &\quad - \frac{1}{1-\sigma} PB\Sigma_2^{-1} B^T P - \frac{1}{1-\sigma}\|\chi\|^2 P^2 - \frac{1}{1-\sigma} L\big(T_H^T T_H + \Sigma_2\big)L > 0, \end{aligned}\tag{9}$$

where $L = \mathrm{diag}(L_i)$, $i = 1, 2, \dots, n$, $\chi = (\chi_1, \chi_2, \dots, \chi_n)^T$.

Proof. Consider the Lyapunov functional

$$V(t) = x^T(t)Px(t) + \sum_{i=1}^n \sum_{j=1}^n \int_{t-\tau_j(t)}^{t} q_{ij} f_j^2\big(x(s)\big)\,ds, \tag{10}$$

where $Q = (q_{ij})_{n\times n} = T_H^T T_H + \Sigma_2$.

Now we compute the derivative of along the trajectories of (5), giving

$$\begin{aligned}\dot V(t) &\leqslant -x^T(t)\big(PC + C^T P\big)x(t) + 2x^T(t)PAf\big(x(t)\big) \\ &\quad + 2x^T(t)PBf\big(x\big(t-\tau(t)\big)\big) + 2x^T(t)P\Gamma^T T_H f\big(x\big(t-\tau(t)\big)\big) \\ &\quad + f^T\big(x(t)\big)Qf\big(x(t)\big) - (1-\sigma)f^T\big(x\big(t-\tau(t)\big)\big)Qf\big(x\big(t-\tau(t)\big)\big). \end{aligned}\tag{11}$$

By Lemma 1, we get

$$\begin{aligned}&2x^T(t)PAf\big(x(t)\big) \leqslant \frac{1}{\varepsilon} x^T(t)PA\Sigma_1^{-1}A^T Px(t) + \varepsilon f^T\big(x(t)\big)\Sigma_1 f\big(x(t)\big), \\ &2x^T(t)PBf\big(x\big(t-\tau(t)\big)\big) \\ &\quad \leqslant \frac{1}{1-\sigma} x^T(t)PB\Sigma_2^{-1}B^T Px(t) + (1-\sigma)f^T\big(x\big(t-\tau(t)\big)\big)\Sigma_2 f\big(x\big(t-\tau(t)\big)\big), \\ &2x^T(t)P\Gamma^T T_H f\big(x\big(t-\tau(t)\big)\big) \\ &\quad \leqslant \frac{1}{1-\sigma} x^T(t)P\Gamma^T \Gamma Px(t) + (1-\sigma)f^T\big(x\big(t-\tau(t)\big)\big)T_H^T T_H f\big(x\big(t-\tau(t)\big)\big). \end{aligned}$$

Since $\Gamma^T\Gamma = \|\zeta\|^2 I$ and $\|\zeta\| \leqslant \|\chi\|$, it is clear that

$$x^T(t)P\Gamma^T\Gamma Px(t) \leqslant \|\chi\|^2 x^T(t)P^2x(t). \tag{12}$$

And since $Q = T_H^T T_H + \Sigma_2$, it follows that

$$\begin{aligned}\dot V(t) \leqslant{}& -x^T(t)\big(PC + C^T P\big)x(t) + \frac{1}{\varepsilon}x^T(t)PA\Sigma_1^{-1}A^T Px(t)\\ &+ \varepsilon f^T\big(x(t)\big)\Sigma_1 f\big(x(t)\big) + \frac{1}{1-\sigma}x^T(t)PB\Sigma_2^{-1}B^T Px(t)\\ &+ \frac{1}{1-\sigma}\|\chi\|^2 x^T(t)P^2x(t)\\ &+ \frac{1}{1-\sigma}f^T\big(x(t)\big)\big(T_H^T T_H + \Sigma_2\big)f\big(x(t)\big).\end{aligned} \tag{13}$$

By (3), we have

$$\begin{aligned}\dot V(t) &\leqslant x^T(t)\bigg[-PC - C^T P + \frac{1}{\varepsilon}PA\Sigma_1^{-1}A^T P + \varepsilon L\Sigma_1 L\\ &\quad + \frac{1}{1-\sigma}PB\Sigma_2^{-1}B^T P + \frac{1}{1-\sigma}\|\chi\|^2P^2 + \frac{1}{1-\sigma}L\big(T_H^T T_H + \Sigma_2\big)L\bigg]x(t)\\ &\equiv -x^T(t)\Omega x(t).\end{aligned} \tag{14}$$

Then we have $\dot V(t) < 0$ when $\Omega > 0$, i.e., the inequality (9) holds, which completes the proof of the theorem. □

By constructing another Lyapunov functional, we can obtain the following result.

Theorem 2. *The origin of system* (5) *is asymptotically stable if condition* (H_1) *is satisfied and moreover there exist a positive definite symmetric matrix Σ_1 and a positive diagonal matrix $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$, such that*

$$\begin{bmatrix} (DC + CD)L^{-1} - DA - A^T D - \frac{1}{1-\sigma}(T_H^T T_H + \Sigma_1) & DB & D\\ B^T D & \Sigma_1 & 0\\ D & 0 & \frac{1}{\|\chi\|^2}I\end{bmatrix} > 0. \tag{15}$$

Proof. Consider the following Lyapunov functional:

$$V(t) = 2\sum_{i=1}^n d_i\int_0^{x_i} f_i(s)\,ds + \frac{1}{1-\sigma}\sum_{i=1}^n\sum_{j=1}^n\int_{t-\tau_j(t)}^t q_{ij}f_j^2\big(x(s)\big)\,ds, \tag{16}$$

where $Q = (q_{ij})_{n\times n} = T_H^T T_H + \Sigma_1$, $d_i > 0$, $i = 1, 2, \ldots, n$.

Define

$$G(x) = \min\left\{\min\left\{\int_0^{x_i} f_i(\theta)\,d\theta, \int_0^{-x_i} f_i(\theta)\,d\theta\right\},\ i = 1, 2, \ldots, n\right\},$$

which satisfies

$$G(r) > 0,\quad r > 0,\qquad G(r) \to +\infty,\quad r \to +\infty,$$

and $G(0) = 0$, $G(x) = G(|x|)$, for $x \in R_+^n$.

We have

$$V\big(x(t)\big) \geqslant 2\sum_{i=1}^{n} d_i \int_0^{x_i} f_i(s)\,ds \geqslant 2\lambda_m(D)G\big(|x|\big), \tag{17}$$

which gives a lower bound by a positive radially unbound function.

It is easy to verify that

$$2\lambda_m(D)G\big(|x|\big) \leqslant V\big(x(t)\big) \leqslant \left[2q\lambda_M(DL) + \frac{\tau(t)}{1-\sigma}q\lambda_M(LQL)\right]\big\|x(t)\big\|^2, \quad q > 1. \tag{18}$$

The derivative of this functional along the solution of system (5) is

$$\begin{aligned}\dot V(t) \leqslant{}& f^T\big(x(t)\big)D\big[-Cx(t) + Af\big(x(t)\big) + \big(B + \Gamma^T T_H\big)f\big(x\big(t-\tau(t)\big)\big)\big] \\ &+ \big[-Cx(t) + Af\big(x(t)\big) + \big(B + \Gamma^T T_H\big)f\big(x\big(t-\tau(t)\big)\big)\big]^T Df\big(x(t)\big) \\ &+ \frac{1}{1-\sigma}f^T\big(x(t)\big)Qf\big(x(t)\big) - f^T\big(x\big(t-\tau(t)\big)\big)Qf\big(x\big(t-\tau(t)\big)\big).\end{aligned} \tag{19}$$

Then it follows from Lemma 1 that

$$\begin{aligned}& f^T\big(x(t)\big)DBf\big(x\big(t-\tau(t)\big)\big) + f^T\big(x\big(t-\tau(t)\big)\big)B^T Df\big(x(t)\big) \\ &\quad\leqslant f^T\big(x(t)\big)DB\Sigma_1^{-1}B^T Df\big(x(t)\big) + f^T\big(x\big(t-\tau(t)\big)\big)\Sigma_1 f\big(x\big(t-\tau(t)\big)\big),\end{aligned} \tag{20}$$

$$\begin{aligned}& f^T\big(x(t)\big)D\Gamma^T T_H f\big(x\big(t-\tau(t)\big)\big) + f^T\big(x\big(t-\tau(t)\big)\big)T_H^T \Gamma Df\big(x(t)\big) \\ &\quad\leqslant f^T\big(x(t)\big)D\Gamma^T\Gamma Df\big(x(t)\big) + f^T\big(x\big(t-\tau(t)\big)\big)T_H^T T_H f\big(x\big(t-\tau(t)\big)\big) \\ &\quad\leqslant \|\chi\|^2 f^T\big(x(t)\big)D^2 f\big(x(t)\big) + f^T\big(x\big(t-\tau(t)\big)\big)T_H^T T_H f\big(x\big(t-\tau(t)\big)\big).\end{aligned} \tag{21}$$

To this end, substituting (20), (21) into (19) and for $Q = T_H^T T_H + \Sigma_1$, we have

$$\begin{aligned}\dot V(t) &\leqslant f^T\big(x(t)\big)\Big[-DCL^{-1} - CDL^{-1} + DA + A^T D + DB\Sigma_1^{-1}B^T D \\ &\qquad + \|\chi\|^2 D^2 + \frac{1}{1-\sigma}\big(T_H^T T_H + \Sigma_1\big)\Big]f\big(x(t)\big) \\ &\equiv -f^T\big(x(t)\big)\Omega f\big(x(t)\big).\end{aligned} \tag{22}$$

Then we have $\dot V(t) < 0$ when $\Omega > 0$. By the Schur complements (see Lemma 2), $\Omega > 0$ if and only if inequality (15) holds, which completes the proof of the theorem. □

Theorem 3. *The origin of system* (5) *is the unique equilibrium point and it is globally asymptotically stable if the condition* (H_1) *is satisfied and there exist a positive definite matrix P, a positive diagonal matrix D and a positive constant β such that*

$$\begin{aligned}\Omega = \Big[&-2DCL^{-1} + DA + A^T D + \beta^{-1}DP^{-1}D \\ &+ \frac{1+\|\chi\|^2}{1-\sigma}\beta\lambda_M(P)\big(B^T B + T_H^T T_H\big)\Big] < 0,\end{aligned} \tag{23}$$

where $L = \operatorname{diag}(L_i)$, $i = 1, 2, \ldots, n$, is a positive diagonal matrix.

Proof. Consider the following Lyapunov functional:

$$\begin{aligned}V(t) &= x^T(t)Cx(t) + 2\alpha\sum_{i=1}^{n} d_i \int_0^{x_i} f_i(s)\,ds \\ &\quad + \alpha\beta\frac{1}{1-\sigma}\int_{t-\tau(t)}^{t} f^T\big(x(s)\big)W^T PWf\big(x(s)\big)\,ds \\ &\quad + \frac{1}{1-\sigma}\int_{t-\tau(t)}^{t} f^T\big(x(s)\big)W^T Wf\big(x(s)\big)\,ds, \end{aligned}\tag{24}$$

where $W = B + \Gamma^T T_H$.

Then the derivative of this functional along the solution of system (5) is

$$\begin{aligned}\dot V(t) &\leqslant -x^T(t)C^2x(t) - x^T(t)C^2x(t) + 2x^T(t)CAf\big(x(t)\big) \\ &\quad + 2x^T(t)CWf\big(x\big(t-\tau(t)\big)\big) - 2\alpha f^T\big(x(t)\big)DCx(t) \\ &\quad + 2\alpha f^T\big(x(t)\big)DAf\big(x(t)\big) + 2\alpha f^T\big(x(t)\big)DWf\big(x\big(t-\tau(t)\big)\big) \\ &\quad + \alpha\beta\frac{1}{1-\sigma}f^T\big(x(t)\big)W^T PWf\big(x(t)\big) + \frac{1}{1-\sigma}f^T\big(x(t)\big)W^T Wf\big(x(t)\big) \\ &\quad - \alpha\beta f^T\big(x\big(t-\tau(t)\big)\big)W^T PWf\big(x\big(t-\tau(t)\big)\big) \\ &\quad - f^T\big(x\big(t-\tau(t)\big)\big)W^T Wf\big(x\big(t-\tau(t)\big)\big). \end{aligned}\tag{25}$$

Adding and subtracting $\alpha\beta^{-1}f^T(x(t))DP^{-1}Df(x(t))$ in the above equation results in

$$\begin{aligned}\dot V(t) &\leqslant -x^T(t)C^2x(t) - x^T(t)C^2x(t) + 2x^T(t)CAf\big(x(t)\big) \\ &\quad + 2x^T(t)CWf\big(x\big(t-\tau(t)\big)\big) - 2\alpha f^T\big(x(t)\big)DCx(t) \\ &\quad + 2\alpha f^T\big(x(t)\big)DAf\big(x(t)\big) + 2\alpha f^T\big(x(t)\big)DWf\big(x\big(t-\tau(t)\big)\big) \\ &\quad + \alpha\beta\frac{1}{1-\sigma}f^T\big(x(t)\big)W^T PWf\big(x(t)\big) + \frac{1}{1-\sigma}f^T\big(x(t)\big)W^T Wf\big(x(t)\big) \\ &\quad + \alpha\beta^{-1}f^T\big(x(t)\big)DP^{-1}Df\big(x(t)\big) - \alpha\beta^{-1}f^T\big(x(t)\big)DP^{-1}Df\big(x(t)\big) \\ &\quad - \alpha\beta f^T\big(x\big(t-\tau(t)\big)\big)W^T PWf\big(x\big(t-\tau(t)\big)\big) \\ &\quad - f^T\big(x\big(t-\tau(t)\big)\big)W^T Wf\big(x\big(t-\tau(t)\big)\big). \end{aligned}\tag{26}$$

Using the inequality technique, we have

$$\begin{aligned}&-x^T(t)C^2x(t) + 2x^T(t)CAf\big(x(t)\big) \\ &\quad = -\big(Cx(t) - Af\big(x(t)\big)\big)^T\big(Cx(t) - Af\big(x(t)\big)\big) + f^T\big(x(t)\big)A^T Af\big(x(t)\big), \end{aligned}\tag{27}$$

$$\begin{aligned}&-x^T(t)C^2x(t) + 2x^T(t)CWf\big(x\big(t-\tau(t)\big)\big) \\ &\quad = -\big(Cx(t) - Wf\big(x\big(t-\tau(t)\big)\big)\big)^T\big(Cx(t) - Wf\big(x\big(t-\tau(t)\big)\big)\big) \\ &\qquad + f^T\big(x\big(t-\tau(t)\big)\big)W^T Wf\big(x\big(t-\tau(t)\big)\big), \end{aligned}\tag{28}$$

$$\begin{aligned}
&-\alpha\beta^{-1} f^T\big(x(t)\big)DP^{-1}Df\big(x(t)\big) + 2\alpha f^T\big(x(t)\big)DWf\big(x\big(t-\tau(t)\big)\big)\\
&\quad = -\alpha\big[\beta^{-1/2}DP^{-1/2}f\big(x(t)\big) - \beta^{1/2}P^{1/2}Wf\big(x\big(t-\tau(t)\big)\big)\big]^T\\
&\qquad \times \big[\beta^{-1/2}DP^{-1/2}f\big(x(t)\big) - \beta^{1/2}P^{1/2}Wf\big(x\big(t-\tau(t)\big)\big)\big]\\
&\qquad + \alpha\beta f^T\big(x\big(t-\tau(t)\big)\big)W^TPWf\big(x\big(t-\tau(t)\big)\big). \qquad (29)
\end{aligned}$$

Since the first terms of the above equations are non-positive, we can write the following inequalities:

$$-x^T(t)C^2x(t) + 2x^T(t)CAf\big(x(t)\big) \leqslant f^T\big(x(t)\big)A^TAf\big(x(t)\big), \qquad (30)$$

$$\begin{aligned}
&-x^T(t)C^2x(t) + 2x^T(t)CWf\big(x\big(t-\tau(t)\big)\big)\\
&\quad \leqslant f^T\big(x\big(t-\tau(t)\big)\big)W^TWf\big(x\big(t-\tau(t)\big)\big), \qquad (31)
\end{aligned}$$

$$\begin{aligned}
&-\alpha\beta^{-1} f^T\big(x(t)\big)DP^{-1}Df\big(x(t)\big) + 2\alpha f^T\big(x(t)\big)DWf\big(x\big(t-\tau(t)\big)\big)\\
&\quad \leqslant \alpha\beta f^T\big(x\big(t-\tau(t)\big)\big)W^TPWf\big(x\big(t-\tau(t)\big)\big). \qquad (32)
\end{aligned}$$

From (3), we obtain the following:

$$-2\alpha f^T\big(x(t)\big)DCx(t) \leqslant -2\alpha f^T\big(x(t)\big)DCL^{-1}f\big(x(t)\big). \qquad (33)$$

By substituting inequalities (30)–(33) into (26), we obtain

$$\begin{aligned}
\dot{V}(t) &\leqslant f^T\big(x(t)\big)A^TAf\big(x(t)\big) - 2\alpha f^T\big(x(t)\big)DCL^{-1}f\big(x(t)\big)\\
&\quad + 2\alpha f^T\big(x(t)\big)DAf\big(x(t)\big) + \alpha\beta\frac{1}{1-\sigma}f^T\big(x(t)\big)W^TPWf\big(x(t)\big)\\
&\quad + \frac{1}{1-\sigma}f^T\big(x(t)\big)W^TWf\big(x(t)\big) + \alpha\beta^{-1}f^T\big(x(t)\big)DP^{-1}Df\big(x(t)\big)\\
&\leqslant f^T\big(x(t)\big)A^TAf\big(x(t)\big) - 2\alpha f^T\big(x(t)\big)DCL^{-1}f\big(x(t)\big)\\
&\quad + 2\alpha f^T\big(x(t)\big)DAf\big(x(t)\big) + \alpha\beta^{-1}f^T\big(x(t)\big)DP^{-1}Df\big(x(t)\big)\\
&\quad + \frac{1}{1-\sigma}\big(\alpha\beta\lambda_M(P)+1\big)f^T\big(x(t)\big)W^TWf\big(x(t)\big). \qquad (34)
\end{aligned}$$

Since

$$\begin{aligned}
W^TW &= \big(B+\Gamma^TT_H\big)^T\big(B+\Gamma^TT_H\big) = B^TB + B^T\Gamma^TT_H + T_H^T\Gamma B + T_H^T\Gamma\Gamma^TT_H\\
&\leqslant \big(1+\|\chi\|^2\big)B^TB + \big(1+\|\chi\|^2\big)T_H^TT_H = \big(1+\|\chi\|^2\big)\big(B^TB+T_H^TT_H\big), \qquad (35)
\end{aligned}$$

it follows from (23) that

$$\begin{aligned}
\dot{V}(t) &\leqslant f^T\big(x(t)\big)\big[A^TA + \big(B^TB+T_H^TT_H\big)\big]f\big(x(t)\big)\\
&\quad + \alpha f^T\big(x(t)\big)\bigg[-2DCL^{-1} + DA + A^TD + \beta^{-1}DP^{-1}D\\
&\quad + \frac{1+\|\chi\|^2}{1-\sigma}\beta\lambda_M(P)\big(B^TB+T_H^TT_H\big)\bigg]f\big(x(t)\big)\\
&\leqslant \lambda_M\big(A^TA+B^TB+T_H^TT_H\big)f^T\big(x(t)\big)f\big(x(t)\big) - \alpha\lambda_m(-\Omega)f^T\big(x(t)\big)f\big(x(t)\big)\\
&= \big[\lambda_M\big(A^TA+B^TB+T_H^TT_H\big) - \alpha\lambda_m(-\Omega)\big]\big\|f\big(x(t)\big)\big\|_2^2. \qquad (36)
\end{aligned}$$

The choice

$$\alpha > \frac{\lambda_M(A^T A + B^T B + T_H^T T_H)}{\lambda_m(-\Omega)} > 0$$

ensures that $\dot{V}(t) < 0$, $\forall f(x(t)) \neq 0$. Note that $f(x(t)) \neq 0$ implies that $x(t) \neq 0$. Now let $f(x(t)) = 0$ and $x(t) \neq 0$. In this case, $\dot{V}(t)$ is in the following form:

$$\begin{aligned} \dot{V}(t) \leqslant{} & -x^T(t)C^2x(t) - x^T(t)C^2x(t) + 2x^T(t)CWf\big(x\big(t-\tau(t)\big)\big) \\ & - \alpha\beta f^T\big(x\big(t-\tau(t)\big)\big)W^T PWf\big(x\big(t-\tau(t)\big)\big) \\ & - f^T\big(x\big(t-\tau(t)\big)\big)W^T Wf\big(x\big(t-\tau(t)\big)\big). \end{aligned} \tag{37}$$

Since

$$W^T PW > 0, \qquad \beta > 0, \qquad \alpha > 0,$$

$\dot{V}(t)$ satisfies

$$\begin{aligned} \dot{V}(t) \leqslant{} & -x^T(t)C^2x(t) - x^T(t)C^2x(t) + 2x^T(t)CWf\big(x\big(t-\tau(t)\big)\big) \\ & - f^T\big(x\big(t-\tau(t)\big)\big)W^T Wf\big(x\big(t-\tau(t)\big)\big). \end{aligned} \tag{38}$$

Using (33) in the above inequality, we obtain

$$\dot{V}(t) \leqslant -x^T(t)C^2x(t).$$

Since C is a positive diagonal matrix, we can conclude that

$$\dot{V}(t) \leqslant -x^T(t)C^2x(t) < 0, \quad \forall x(t) \neq 0.$$

Now consider the case where $f(x(t)) = x(t) = 0$. In this case, $\dot{V}(t)$ takes the form:

$$\begin{aligned} \dot{V}(t) ={} & -\alpha\beta f^T\big(x\big(t-\tau(t)\big)\big)W^T PWf\big(x\big(t-\tau(t)\big)\big) \\ & - f^T\big(x\big(t-\tau(t)\big)\big)W^T Wf\big(x\big(t-\tau(t)\big)\big). \end{aligned}$$

Since $W^T PW > 0$, it directly follows that

$$\dot{V}(t) < 0, \quad \forall f\big(x(t-\tau)\big) \neq 0.$$

Hence, we have proved that $\dot{V}(t) = 0$ if and only if $f(x(t)) = x(t) = f(x(t-\tau)) = 0$, otherwise $\dot{V}(t) = 0$. On the other hand, $V(t)$ is radially unbounded, that is $V(t) \to \infty$ as $\|x(t)\| \to \infty$. Thus, from standard Lyapunov stability theorems (see, e.g., [19, Corollary 3.2, Chapter 3]), it follows that the origin of system (2) is GAS. □

Theorem 4. *Under the assumption given by* (H_1), *the neural network model* (1) *is globally asymptotically stable, if the following condition hold*:

$$2r^2 - 2r\|A\|_2 - \|W\|_2^2 - \frac{r^2}{1-\sigma} > 0, \tag{39}$$

where $r = \min_{1 \leqslant i \leqslant n}(c_i/L_i)$, $W = B + \Gamma^T T_H$.

Proof. Construct the following positive definite Lyapunov functional:

$$V(t) = x^T(t)x(t) + 2\alpha r \sum_{i=1}^{n} d_i \int_0^{x_i} f_i(s)\,ds + \big(\alpha r^2 + \beta\big)\frac{1}{1-\sigma} \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} f_i^T\big(x_i(s)\big)\,ds,$$

where α and β are some positive constants to be determined later. Let $W = B + \Gamma^T T_H$, then the time derivative of the functional along the trajectories of system (5) is obtained as follows:

$$\begin{aligned}
\dot{V}(t) \leqslant{} & -2x^T(t)Cx(t) + 2x^T(t)Af\big(x(t)\big) \\
& + 2x^T(t)Wf\big(x\big(t-\tau(t)\big)\big) - 2\alpha r f^T\big(x(t)\big)Cx(t) \\
& + 2\alpha r f^T\big(x(t)\big)Af\big(x(t)\big) + 2\alpha r f^T\big(x(t)\big)Wf\big(x\big(t-\tau(t)\big)\big) \\
& + \frac{\alpha r^2}{1-\sigma} f^T\big(x(t)\big)f\big(x(t)\big) - \alpha r^2 f^T\big(x\big(t-\tau(t)\big)\big)f\big(x\big(t-\tau(t)\big)\big) \\
& + \frac{\beta}{1-\sigma} f^T\big(x(t)\big)f\big(x(t)\big) - \beta f^T\big(x\big(t-\tau(t)\big)\big)f\big(x\big(t-\tau(t)\big)\big).
\end{aligned} \tag{40}$$

We can write the following linear matrix inequalities:

$$\begin{aligned}
& -x^T(t)Cx(t) + 2x^T(t)Af\big(x(t)\big) \\
& \quad = -\big[C^{1/2}x(t) - C^{-1/2}Af\big(x(t)\big)\big]^T\big[C^{1/2}x(t) - C^{-1/2}Af\big(x(t)\big)\big] \\
& \qquad + f^T\big(x(t)\big)A^TC^{-1}Af\big(x(t)\big),
\end{aligned} \tag{41}$$

$$\begin{aligned}
& -x^T(t)Cx(t) + 2x^T(t)Wf\big(x\big(t-\tau(t)\big)\big) \\
& \quad = -\big[C^{1/2}x(t) - C^{-1/2}Wf\big(x\big(t-\tau(t)\big)\big)\big]^T\big[C^{1/2}x(t) - C^{-1/2}Wf\big(x\big(t-\tau(t)\big)\big)\big] \\
& \qquad + f^T\big(x\big(t-\tau(t)\big)\big)W^TC^{-1}Wf\big(x\big(t-\tau(t)\big)\big),
\end{aligned} \tag{42}$$

$$\begin{aligned}
& 2\alpha r f^T\big(x(t)\big)Wf\big(x\big(t-\tau(t)\big)\big) - \alpha r^2 f^T\big(x\big(t-\tau(t)\big)\big)f\big(x\big(t-\tau(t)\big)\big) \\
& \quad = -\alpha\big[rf\big(x\big(t-\tau(t)\big)\big) - Wf\big(x(t)\big)\big]^T\big[rf\big(x\big(t-\tau(t)\big)\big) - Wf\big(x(t)\big)\big] \\
& \qquad + \alpha f^T\big(x(t)\big)W^TWf\big(x(t)\big).
\end{aligned} \tag{43}$$

Since the first terms of the above equations are non-positive, we can write the following inequalities:

$$-x^T(t)Cx(t) + 2x^T(t)Af\big(x(t)\big) \leqslant f^T\big(x(t)\big)A^TC^{-1}Af\big(x(t)\big), \tag{44}$$

$$\begin{aligned}
& -x^T(t)Cx(t) + 2x^T(t)Wf\big(x\big(t-\tau(t)\big)\big) \\
& \quad \leqslant f^T\big(x\big(t-\tau(t)\big)\big)W^TC^{-1}Wf\big(x\big(t-\tau(t)\big)\big),
\end{aligned} \tag{45}$$

$$\begin{aligned}
& 2\alpha r f^T\big(x(t)\big)Wf\big(x\big(t-\tau(t)\big)\big) - \alpha r^2 f^T\big(x\big(t-\tau(t)\big)\big)f\big(x\big(t-\tau(t)\big)\big) \\
& \quad \leqslant \alpha f^T\big(x(t)\big)W^TWf\big(x(t)\big).
\end{aligned} \tag{46}$$

Using (44)–(46) in (43) yields:

$$\begin{aligned}
\dot{V}(t) \leqslant{} & -2\alpha r f^T\big(x(t)\big)Cx(t) + 2\alpha r f^T\big(x(t)\big)Af\big(x(t)\big) + f^T\big(x(t)\big)A^TC^{-1}Af\big(x(t)\big) \\
& + f^T\big(x\big(t-\tau(t)\big)\big)W^TC^{-1}Wf\big(x\big(t-\tau(t)\big)\big) + \alpha f^T\big(x(t)\big)W^TWf\big(x(t)\big) \\
& + \frac{\alpha r^2}{1-\sigma} f^T\big(x(t)\big)f\big(x(t)\big) + \frac{\beta}{1-\sigma} f^T\big(x(t)\big)f\big(x(t)\big) \\
& - \beta f^T\big(x\big(t-\tau(t)\big)\big)f2\big(x\big(t-\tau(t)\big)\big).
\end{aligned} \tag{47}$$

Using the properties of the activation functions, we obtain

$$\begin{aligned}\dot V(t) \leqslant{}& -2\alpha r\sum_{i=1}^{n}\frac{c_i}{L_i}f_i^2\bigl(x_i(t)\bigr)+2\alpha r\|A\|_2\bigl\|f\bigl(x(t)\bigr)\bigr\|_2^2+\|A\|_2^2\bigl\|C^{-1}\bigr\|_2\bigl\|f\bigl(x(t)\bigr)\bigr\|_2^2\\&+\bigl(\|W\|_2^2\bigl\|C^{-1}\bigr\|_2-\beta\bigr)\bigl\|f\bigl(x\bigl(t-\tau(t)\bigr)\bigr)\bigr\|_2^2+\alpha\|W\|_2^2\bigl\|f\bigl(x(t)\bigr)\bigr\|_2^2\\&+\frac{\alpha r^2}{1-\sigma}\bigl\|f\bigl(x(t)\bigr)\bigr\|_2^2+\frac{\beta}{1-\sigma}\bigl\|f\bigl(x(t)\bigr)\bigr\|_2^2.\end{aligned}\tag{48}$$

Since $r=\min_{1\leqslant i\leqslant n}(c_i/L_i)$, we have

$$-2\alpha r\sum_{i=1}^{n}\frac{c_i}{L_i}f_i^2\bigl(x_i(t)\bigr)\leqslant -2\alpha r^2\sum_{i=1}^{n}f_i^2\bigl(x_i(t)\bigr)=-2\alpha r^2\bigl\|f\bigl(x(t)\bigr)\bigr\|_2^2.\tag{49}$$

Let $\|W\|_2^2\|C^{-1}\|_2=\beta$. Thus, in the light of above inequality, $\dot V(t)$ can now be written as

$$\begin{aligned}\dot V(t)\leqslant{}&-\alpha\left(2r^2-2r\|A\|_2-\|W\|_2^2-\frac{r^2}{1-\sigma}\right)\bigl\|f\bigl(x(t)\bigr)\bigr\|_2^2\\&+\left(\|A\|_2^2\bigl\|C^{-1}\bigr\|_2+\frac{1}{1-\sigma}\|W\|_2^2\bigl\|C^{-1}\bigr\|_2\right)\bigl\|f\bigl(x(t)\bigr)\bigr\|_2^2.\end{aligned}\tag{50}$$

Since $2r^2-2r\|A\|_2-\|W\|_2^2-\frac{r^2}{1-\sigma}>0$, the choice

$$\alpha>\frac{\|A\|_2^2\|C^{-1}\|_2+\frac{1}{1-\sigma}\|W\|_2^2\|C^{-1}\|_2}{2r^2-2r\|A\|_2-\|W\|_2^2-\frac{r^2}{1-\sigma}}>0$$

ensures that $\dot V(t)$ is negative definite for all $f(x(t))\neq 0$. (It should be noted here that $f(x(t))\neq 0$ implies that $x(t)\neq 0$.) Now consider the case where $f(x(t))=0$ and $x(t)\neq 0$. In this case, $\dot V(t)$ satisfies

$$\begin{aligned}\dot V(t)\leqslant{}&-2x^T(t)Cx(t)+2x^T(t)Wf\bigl(x\bigl(t-\tau(t)\bigr)\bigr)-\alpha r^2f^T\bigl(x\bigl(t-\tau(t)\bigr)\bigr)f\bigl(x\bigl(t-\tau(t)\bigr)\bigr)\\&-\beta f^T\bigl(x\bigl(t-\tau(t)\bigr)\bigr)f\bigl(x\bigl(t-\tau(t)\bigr)\bigr)\\\leqslant{}&-2x^T(t)Cx(t)+2x^T(t)Wf\bigl(x\bigl(t-\tau(t)\bigr)\bigr)\\&-\beta f^T\bigl(x\bigl(t-\tau(t)\bigr)\bigr)f\bigl(x\bigl(t-\tau(t)\bigr)\bigr).\end{aligned}\tag{51}$$

Since

$$-2x^T(t)Cx(t)+2x^T(t)Wf\bigl(x\bigl(t-\tau(t)\bigr)\bigr)-\beta f^T\bigl(x\bigl(t-\tau(t)\bigr)\bigr)f\bigl(x\bigl(t-\tau(t)\bigr)\bigr)\leqslant 0,$$

then

$$\dot V(t)\leqslant -\beta x^T(t)Cx(t).$$

It follows that $\dot V(t)$ is negative definite for all $x(t)\neq 0$ with $f(x(t))=0$. Finally, consider the case where $f(x(t))=0$ and $x(t)=0$. This case implies that

$$\dot V(t)\leqslant -\alpha r^2f^T\bigl(x\bigl(t-\tau(t)\bigr)\bigr)f\bigl(x\bigl(t-\tau(t)\bigr)\bigr)-\beta f^T\bigl(x\bigl(t-\tau(t)\bigr)\bigr)f\bigl(x\bigl(t-\tau(t)\bigr)\bigr).\tag{52}$$

Obviously, $\dot V(t)$ is negative definite for all $f(x(t-\tau(t)))\neq 0$. Hence, it follows that $\dot V(t)=0$ if and only if $x(t)=f(x(t))=f(x(t-\tau(t)))=0$, otherwise $\dot V(t)<0$. Moreover, $V(t)$ is radially unbounded since $V(t) \to \infty$ as $t \to \infty$. Thus, it can be concluded that the origin system (5), or equivalently the equilibrium point of system (1) is globally asymptotically stable. $\square$

If we choose $\Sigma_2 = I$, then the condition in Theorem 1 becomes

Corollary 1. *The origin of system* (5) *is asymptotically stable if condition* (H_1) *is satisfied and moreover there exist positive definite symmetric matrices* P, Σ_1 *and a scalar* $\varepsilon > 0$ *such that*

$$\begin{aligned}&-PC - C^T P + \frac{1}{\varepsilon} PA\Sigma_1^{-1}A^T P + \varepsilon L\Sigma_1 L\\&\quad + \frac{1}{1-\sigma} PB\Sigma_2^{-1}B^T P + \frac{1}{1-\sigma}\|\chi\|^2 P^2 + \frac{1}{1-\sigma} L\big(T_H^T T_H + I\big)L < 0,\end{aligned} \tag{53}$$

where $L = \mathrm{diag}(L_i)$, $i = 1, 2, \ldots, n$.

Remark 1. In Corollary 1, while the case for time-varying delays reduces to constant delays, i.e., $\sigma = 0$, we can see that Corollary 2 given in [15] can be considered as the special cases of the result.

If we choose $\Sigma_1 = \Sigma_2 = P = I$, then the condition in Theorem 1 becomes

Corollary 2. *The origin of system* (5) *is asymptotically stable if condition* (H_1) *is satisfied and a scalar* $\varepsilon > 0$ *such that*

$$-2C + \frac{1}{\varepsilon} AA^T + \varepsilon L^2 + \frac{1}{1-\sigma} BB^T + \frac{1}{1-\sigma}\|\chi\|^2 I + \frac{1}{1-\sigma} L\big(T_H^T T_H + I\big)L < 0, \tag{54}$$

where $L = \mathrm{diag}(L_i)$, $i = 1, 2, \ldots, n$.

Let $L = C = I$. If we choose $\Sigma_1 = \Sigma_2 = I$, $P = \frac{1}{\delta} I$ with $\delta > 0$, then the condition in Theorem 1 becomes

Corollary 3. *The origin of system* (5) *is asymptotically stable if condition* (H_1) *is satisfied and a scalar* $\varepsilon > 0$ *such that*

$$\begin{aligned}&(2-\varepsilon)I - \frac{1}{1-\sigma}\big(T_H^T T_H + I\big) + \frac{1}{\delta^2\varepsilon} AA^T + \frac{1}{\delta^2(1-\sigma)} BB^T\\&\quad + \frac{1}{\delta^2(1-\sigma)}\|\chi\|^2 I < 0.\end{aligned} \tag{55}$$

In the above Theorem 2, if $C = D = \Sigma_1 = I$, then we have the following corollary.

Corollary 4. *The origin of system* (5) *is asymptotically stable if condition* (H_1) *is satisfied and such that*

$$\begin{bmatrix} 2I - A - A^T - \frac{1}{1-\sigma}(T_H^T T_H + I) & B & I \\ B^T & I & 0 \\ I & 0 & \frac{1}{\|\chi\|^2} I \end{bmatrix} > 0. \tag{56}$$

Corollary 5. *Under the assumption given by* (H_1), *the neural network model* (1) *is globally asymptotically stable, if the following condition hold*:

$$2\|A\|_2 + \left\|B + \Gamma^T T_H\right\|_2^2 < \frac{2\sigma - 1}{1 - \sigma}. \tag{57}$$

Remark 2. As we know, the architecture with high-order interactions [20,21] have been successively introduced to design neural networks which have stronger approximation property, faster convergence rate, greater storage capacity, and higher fault tolerance than lower-order neural networks. So the case studied in this paper is more general. Let $T_H \equiv 0$, the high-order Hopfield-type neural network with time-varying delays becomes the lower-order Hopfield neural networks. Moreover, we consider the lower-order Hopfield neural networks with constant time delay, it can easily obtain the same results in [22]. Similarly, according to the analysis in [22], the conditions given [23–25] can be considered as the special cases of the result derived from Theorem 3 with $T_H \equiv 0$.

4. An illustrative example

Consider the following high-order Hopfield-type neural network with time-varying delays:

$$\begin{aligned}\frac{dy_i(t)}{dt} &= -c_i y_i(t) + \sum_{j=1}^{2} a_{ij} g_j\big(y_j(t)\big) + \sum_{j=1}^{2} b_{ij} g_j\big(y_j\big(t - \tau_j(t)\big)\big) \\ &\quad + \sum_{j=1}^{2}\sum_{l=1}^{2} T_{ijl} g_j\big(y_j\big(t - \tau_j(t)\big)\big) g_l\big(y_l\big(t - \tau_l(t)\big)\big) + J_i, \quad i = 1, 2,\end{aligned} \tag{58}$$

where $g_1(y_1) = \tanh(0.53 y_1)$, $g_2(y_2) = \tanh(0.67 y_2)$, $\sigma = 0.6$, $J_1 = 1.5$, $J_2 = 2$,

$$C = \begin{bmatrix} 1.90 & 0 \\ 0 & 1.89 \end{bmatrix}, \qquad A = \begin{bmatrix} 0.05 & 0.14 \\ 0.20 & 0.31 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.09 & 0.25 \\ 0.21 & 0.45 \end{bmatrix},$$

$$T_1 = \begin{bmatrix} 0.05 & 0.14 \\ -0.06 & 0.05 \end{bmatrix}, \qquad T_2 = \begin{bmatrix} 0.29 & 0.10 \\ 0.23 & -0.14 \end{bmatrix}, \qquad T_3 = \begin{bmatrix} -0.23 & 0.07 \\ 0.09 & -0.02 \end{bmatrix}.$$

Thus we have $L = 0.67$, $\|\chi\|_2 = 0.67$. Now we choose $P = I$, $\varepsilon = 1$, and it is easy to check

$$\begin{aligned}&-PC - C^T P + \frac{1}{\varepsilon} PAA^T P + \varepsilon L^2 + \frac{1}{1-\sigma} PBB^T P \\ &\quad + \frac{1}{1-\sigma}\|\chi\|^2 P^2 + \frac{1}{1-\sigma} L\big(T_H^T T_H + I\big)L = \begin{bmatrix} -0.9038 & 0.3806 \\ 0.3806 & -0.3284 \end{bmatrix} < 0.\end{aligned}$$

It follows from Corollary 2 that the equilibrium point of the system (58) is globally asymptotically stable.

To use Theorem 2, we let $D = \Sigma_1 = I$, and so

$$\begin{aligned}&-DCL^{-1} - CDL^{-1} + DA + A^T D + DB\Sigma_1^{-1} B^T D \\ &\quad + \|\chi\|^2 D^2 + \frac{1}{1-\sigma}\big(T_H^T T_H + \Sigma_1\big) = \begin{bmatrix} -4.0410 & 0.4686 \\ 0.4686 & -1.8138 \end{bmatrix} < 0.\end{aligned}$$

Therefore, from Theorem 2 that the equilibrium point of the system (58) is globally asymptotically stable.

To use Theorem 4, we have

$$2r^2 - 2r\|A\|_2 - \|W\|_2^2 - \frac{r^2}{1-\sigma} = 0.0860 > 0, \tag{59}$$

thus the equilibrium point of the system (58) is globally asymptotically stable.

From the above results, the example demonstrate the effectiveness of the conditions.

5. Conclusions

In this paper, some new criteria for global asymptotic stability of high-order Hopfield-type neural networks with time-varying delays have been derived. Our results have generalized some existing results reported in the literature for both cases with constant and time-varying delays. Moreover, the LMI-based criteria obtained here are computationally more flexible and more efficient than many other existing criteria. So the results extend and improve the earlier publications. An example is also worked through to illustrate our results.

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